Maclaurinův polynom 3. stupně pro $f(x) = \arcsin x$

$$T_3(x) = f(0) + \frac{f'(0)\cdot x}{1!} + \frac{f''(0)\cdot x^2}{2!} + \frac{f'''(0)\cdot x^3}{3!}$$

$x_0 = 0$

$$f(0) = \arcsin 0 = 0$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} \longrightarrow f'(0) = 1$$

$$f''(x) = \left(\frac{1}{\sqrt{1-x^2}}\right)' = \left(\left(1-x^2\right)^{-\frac{1}{2}}\right)' = -\frac{1}{2}\left(1-x^2\right)^{-\frac{3}{2}}\cdot(-2x) = \frac{x}{\sqrt{(1-x^2)^3}} \to f''(0) = 0$$

$$\left(1-x^2\right)^{\frac{3}{2}}$$

$$f'''(x) = \frac{(x)'\cdot\sqrt{(1-x^2)^3} - x\cdot\left(\sqrt{(1-x^2)^3}\right)'}{\left(\sqrt{(1-x^2)^3}\right)^2} =$$

$$= \frac{1\cdot\sqrt{(1-x^2)^3} - x\cdot\frac{3}{2}\left(1-x^2\right)^{\frac{1}{2}}\cdot(-2x)}{(1-x^2)^3} \longrightarrow f'''(0) = 1$$

$$T_3(x) = 0 + 1\cdot x + 0\cdot\frac{x^2}{2!} + \frac{1\cdot x^3}{3!} = \boxed{x + \frac{x^3}{6}}$$

Taylorův polynom 3. stupně pro $f(x) = \operatorname{arctg}\frac{1}{x}$ v $x_0 = 1$

$$T_3(x) = f(1) + \frac{f'(1)(x-1)}{1!} + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!}$$

$$f(1) = \operatorname{arctg} 1 = \frac{\pi}{4}$$

$$f'(x) = \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(\frac{1}{x}\right)' = \frac{1}{1+\frac{1}{x^2}}\cdot(-1)\cdot x^{-2} = \frac{1}{\frac{x^2+1}{x^2}}\cdot\frac{-1}{x^2} = \frac{-1}{x^2+1} \to f'(1) = -\frac{1}{2}$$

$$f''(x) = \left(-\frac{1}{x^2+1}\right)' = \left(-1\cdot\left(x^2+1\right)^{-1}\right)' =$$

$$= -1\cdot(-1)\cdot\left(x^2+1\right)^{-2}\cdot 2x = \frac{2x}{\left(x^2+1\right)^2} \longrightarrow f''(1) = \frac{2}{2^2} = \frac{1}{2}$$

$$f'''(x) = \frac{(2x)'\cdot\left(x^2+1\right)^2 - 2x\cdot\left(\left(x^2+1\right)^2\right)'}{\left(x^2+1\right)^4} =$$

$$= \frac{2\left(x^2+1\right)^2 - 2x\cdot 2\left(x^2+1\right)\cdot 2x}{\left(x^2+1\right)^4} \longrightarrow f'''(1) = \frac{2\cdot 2^2 - 2\cdot 2\cdot 2\cdot 2}{2^4}$$

$$= \frac{8-16}{16} = \frac{-8}{16} = -\frac{1}{2}$$

$$\boxed{T_3(x) = \frac{\pi}{4} + \left(-\frac{1}{2}\right)(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{12}(x-1)^3}$$

$$\frac{\frac{1}{2}}{2!} = \frac{1}{4} \qquad \frac{-\frac{1}{2}}{3!} = \frac{-\frac{1}{2}}{6} = -\frac{1}{12}$$

Určete limity:

a)

$$\lim_{x\to\frac{\pi}{2}}\frac{\sin 2x-2\cos x}{\cos^2 x}=\left[\frac{\sin\overbrace{\left(2\cdot\frac{\pi}{2}\right)}^{\pi}-2\cos\frac{\pi}{2}}{\cos^2\frac{\pi}{2}}=\frac{0-0}{0}=\frac{0}{0}\right]\overset{\text{L.P.}}{=}\lim_{x\to\frac{\pi}{2}}\frac{\cos 2x\cdot 2-2(-\sin x)}{2\cos x\cdot(-\sin x)}=$$

$$=\left[\frac{(\cos\overbrace{2\cdot\frac{\pi}{2}}^{\pi})\cdot 2+2\sin\frac{\pi}{2}}{2\cos\frac{\pi}{2}\cdot(-\sin\frac{\pi}{2})}=\frac{2\cos\pi+2\sin\frac{\pi}{2}}{-2\cos\frac{\pi}{2}\sin\frac{\pi}{2}}=\frac{-2+2}{-2\cdot 0\cdot 1}\right]=\left[\frac{0}{0}\right]\overset{\text{L.P.}}{=}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{2(-\sin 2x)\cdot 2+2\cos x}{-2\underbrace{(-\sin x\cdot\sin x+\cos x\cdot\cos x)}_{(\cos x\sin x)'}}=$$

$$=\frac{-2(\sin\overbrace{2\cdot\frac{\pi}{2}}^{\pi})\cdot 2+2\cos\frac{\pi}{2}}{-2\left(-\sin\frac{\pi}{2}\cdot\sin\frac{\pi}{2}+\cos\frac{\pi}{2}\cdot\cos\frac{\pi}{2}\right)}=\frac{-2\cdot 0\cdot 2+2\cdot 0}{-2\cdot(-1\cdot 1+0\cdot 0)}=\frac{0}{2}=\boxed{0}$$

b)

$$\lim_{x\to 0}\frac{\ln\overbrace{\frac{\sin x}{x}}}{1-\cos x}=\frac{\ln 1}{1-\cos 0}=\frac{0}{1-1}=\left[\frac{0}{0}\right]\ \text{L.P.}$$

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\cos x}{1}=1}$$

$$\overset{\text{L.P.}}{=}\lim_{x\to 0}\frac{\frac{1}{\frac{\sin x}{x}}\cdot\left(\frac{\sin x}{x}\right)'}{0-(-\sin x)}=\lim_{x\to 0}\frac{\frac{1}{\frac{\sin x}{x}}\cdot\frac{\cos x\cdot x-\sin x\cdot 1}{x^2}}{\sin x}=\lim_{x\to 0}\frac{\cos x\cdot x-\sin x}{x\cdot\sin^2 x}=\left[\frac{0}{0}\right]\overset{\text{L.P.}}{=}$$

$$=\lim_{x\to 0}\frac{(-\sin x\cdot x+\cos x)-\cos x}{1\cdot\sin^2 x+x\cdot 2\sin x\cos x}=\lim_{x\to 0}\frac{-\sin x\cdot x}{\sin^2 x+x\cdot 2\sin x\cos x}=\lim_{x\to 0}\frac{-x}{\sin x+2x\cos x}=$$

$$=\left[\frac{0}{0}\right]\overset{\text{L.P.}}{=}\lim_{x\to 0}\frac{-1}{\underbrace{\cos x}_{1}+\underbrace{2\cos x}_{2\cdot 1}+\underbrace{2x\cdot(-\sin x)}_{0}}=\boxed{-\frac{1}{3}}$$

Asymptoty:

① vodorovná pro $f(x) = e^{\frac{1}{x}}$

$$\lim_{x\to\infty} e^{\frac{1}{x}} = e^{\frac{1}{\infty}\to 0} = e^0 = 1$$
$$\lim_{x\to-\infty} e^{\frac{1}{x}} = e^{\frac{1}{-\infty}\to 0} = e^0 = 1$$

$\boxed{y = 1 \text{ pro } x \to \pm\infty}$

---

② svislá pro $f(x) = \dfrac{1}{\ln x}$ → $\ln x \ \ldots \ x \in (0,\infty)$

$\ln x \neq 0$

$x \neq 1$

$D(f) = (0,1) \cup (1,\infty)$

$$\lim_{x\to 1} \frac{1}{\ln x} = \left\langle \begin{array}{l} \lim\limits_{x\to 1^+} \dfrac{1}{\ln x} = \dfrac{1}{0^+} = \infty \\ \lim\limits_{x\to 1^-} \dfrac{1}{\ln x} = \dfrac{1}{0^-} = -\infty \end{array} \right\rangle \quad \boxed{x = 1}$$

---

③ šikmá pro $f(x) = \dfrac{x^2+1}{x}$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\frac{x^2+1}{x}}{x} = \lim_{x\to\infty} \frac{x^2+1}{x^2} = \left[\frac{\infty}{\infty}\right] \overset{\text{L.P.}}{=} \lim_{x\to\infty} \frac{2x}{2x} = 1$$
$$b = \lim_{x\to\infty} \left(f(x) - ax\right) = \lim_{x\to\infty} \left(\frac{x^2+1}{x} - x\right) = \lim_{x\to\infty} \frac{x^2+1-x^2}{x} = \lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

$y = 1 \cdot x + 0 \ \Rightarrow \ \boxed{y = x}$ $\quad \boxed{\begin{array}{l}\text{pro } x \to \infty \\ \text{pro } x \to -\infty\end{array}}$

pozn. $\lim\limits_{x\to-\infty}$ … stejný výpočet